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Random Walks in a Random Field of Decaying Traps

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We study random walks on $\mathbb{Z}^d$ ($d \geqslant 1$) containing traps subject to decay. The initial trap distribution is random. In the course of time, traps decay independently according to a given lifetime distribution. We derive a necessary and sufficient condition under which the walk eventually gets trapped with probability 1. We prove bounds and asymptotic estimates for the survival probability as a function of time and for the average trapping time. These are compared with some well-known results for nondecaying traps.

KEY WORDS: Random walk; decaying random trap field; n-step survival probability; average trapping time; large deviations.

1. INTRODUCTION AND STATEMENT OF RESULTS

The following trapping problem has been studied extensively in the literature. Consider the lattice $\mathbb{Z}^d$ ($d \geqslant 1$) and on it a *random walk* $(X_n)_{n \geqslant 0}$ with single-step probability distribution $p: \mathbb{Z}^d \to [0, 1]$, i.e.,

$$X_0 = 0; \qquad X_{n+1} - X_n \text{ are i.i.d. } \mathbb{Z}^d\text{-valued random variables with}$$
$$P_X(X_{n+1} - X_n = x) = p(x) \qquad (n \geqslant 0, x \in \mathbb{Z}^d)$$

Here P_X denotes the probability measure for the random walk path. We shall assume that (X_n) is aperiodic, meaning that there is no proper sub-

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lattice containing 0 and the support of $p(x)$. Next suppose that on the lattice there is a *random trap field* $(C(x))_{x \in \mathbb{Z}^d}$ with density $c > 0$, i.e.,

$$C(x) \text{ are i.i.d. } \{0, 1\}\text{-valued random variables with}$$
$$P_C(C(x) = 1) = 1 - P_C(C(x) = 0) = c \qquad (x \in \mathbb{Z}^d)$$

Here P_C denotes the probability measure for the random trap field; 1 corresponds to a trap, 0 to a trap-free site. The *trapping time* is defined as

$$T = \inf\{n \geqslant 0\colon C(X_n) = 1\}$$

and the *survival function* as

$$f(n) = P(T > n) \qquad (n \geqslant 0) \tag{1.1}$$

with $P = P_X \times P_C$. The latter expresses the fact that the random walk and the random trap field are assumed to be independent. The problem is to find out how $f(n)$ behaves as a function of n, depending on the choice of d, $p(x)$, and c. In the literature interest has centered on asymptotics of $f(n)$, for small and large n, and moments of T, for small and large c.[1,2]

In this paper we want to examine what happens when the traps are allowed to decay in the course of time, i.e., we want to see how this affects the survival function. To that end we introduce a *decaying random trap field* $(C_n(x))_{n \geqslant 0, x \in \mathbb{Z}^d}$ defined in the following manner. With each site x we associate a *random lifetime* $\tau(x)$ such that

$$\tau(x) \text{ are i.i.d. } \{0, 1,...\}\text{-valued random variables with}$$
$$P_C(\tau(x) > n) = c(n) \qquad (n \geqslant 0,\ x \in \mathbb{Z}^d)$$

and we set

$$C_n(x) = 1 \qquad \text{if} \quad n < \tau(x)$$
$$C_n(x) = 0 \qquad \text{if} \quad n \geqslant \tau(x)$$

So site x is a trap until time $\tau(x)$ and decays to a trap-free site at time $\tau(x)$. The function $c(n)$ is the *trap density function* and will be assumed to be given. Note that $\tau(x) = 0$ means that there is no trap at site x at time $n = 0$ to begin with. We shall therefore sometimes write

$$c(n) = cd(n)$$

where

$$c = c(0) = P_C(\tau(x) > 0)$$
$$d(n) = P_C(\tau(x) > n \mid \tau(x) > 0)$$

with $d(n)$ the *intrinsic trap decay function.* The trapping time now becomes

$$T = \inf\{n \geqslant 0: C_n(X_n) = 1\} \tag{1.2}$$

and the survival function is again given by (1.1) with $P = P_X \times P_C$. Now, of course, P_C describes the decaying random trap field. Our aim will be to study $f(n)$ for various choices of d, $p(x)$, and $c(n)$ and to make a comparison with some well-known results for the nondecaying trap situation [i.e., $c(n) \equiv c$]. The present work is largely introductory in the sense that our main goal is to suggest new questions.

Our starting point is the following formal expression for the survival function, which will be proved in Section 2.1.

Proposition.

$$f(n) = E_X\left(\prod_{0 \leqslant k \leqslant n} [1 - c(k)]^{R(k)} \right) \tag{1.3}$$

with

$$R(k) = 1(X_k \notin \{X_0, X_1, \dots, X_{k-1}\}) \qquad (k \geqslant 0)$$

Here $R(k)$ is the indicator random variable, which is 1 if the random walk hits a new site at time k and is 0 otherwise. The expectation is over the random walk only; the expectation over the decaying random trap field is implicit.

Our first result gives the necessary and sufficient condition under which $f(\infty) = \lim_{n \to \infty} f(n) = 0$, i.e., with probability 1 the random walk eventually gets trapped. The proof will be given in Section 2.2.

Theorem 1. $f(\infty) = 0$ if and only if

$$\sum_{n \geqslant 0} c(n)\, E_X R(n) = \infty \tag{1.4}$$

This gives a complete classification because $E_X R(n)$ can be computed via generating functions. Indeed[3,4]

$$\sum_{n \geqslant 0} z^n E_X R(n) = [(1 - z)\, G(0; z)]^{-1} \tag{1.5}$$

with

$$G(0; z) = (2\pi)^{-d} \int_{(-\pi, \pi]^d} d\theta\, [1 - z\hat{p}(\theta)]^{-1}$$

$$\hat{p}(\theta) = \sum_x e^{i\theta \cdot x} p(x)$$

Thus, for instance, for simple random walk with $p(x) = 1/2d$ if $|x| = 1$ and $p(x) = 0$ otherwise, (1.5) yields[3,4]

$$E_X R(n) \sim \begin{cases} (2/\pi n)^{1/2} & d=1 \\ \pi/\log n & d=2 \quad (n \to \infty) \\ \text{const} & d \geqslant 3 \end{cases}$$

and substitution into (1.4) gives a lower bound for the decay of $c(n)$. Another way of phrasing (1.5) is (ref. 5, p. 36)

$$E_X R(n) = P_X(X_k \neq 0 \text{ for } 0 < k \leqslant n)$$

Hence, for transient random walk (1.4) requires that $\sum_{n \geqslant 0} c(n) = \infty$, i.e., the traps have infinite expected lifetime, while for recurrent random walk the restriction is apparently stronger.

From now on we shall assume that $c(n)$ decays as a power of n with an exponent that is sufficiently small to guarantee $f(\infty) = 0$.

Our second result concerns the rate at which $f(n)$ tends to zero for large n in $d = 1$. (For $d \geqslant 2$ similar but weaker estimates may be obtained.) The proof will be given in Section 2.3.

Theorem 2. Suppose that $d = 1$, $\sum_x xp(x) = 0$, $0 < \sigma^2 = \sum_x x^2 p(x) < \infty$ and $c(n) \sim An^{-\gamma}$ ($n \to \infty$, $A > 0$, $0 < \gamma < 1/2$). Then there exist constants $0 < K_1, K_2 < \infty$ independent of σ, A, and γ such that for large n

$$\exp\left\{ -K_2 \frac{1}{1-2\gamma} (\sigma A)^{2/3} n^{(1-2\gamma)/3} \right\} \leqslant f(n) \leqslant \exp\{ -K_1 (\sigma A)^{2/3} n^{(1-2\gamma)/3} \} \tag{1.6}$$

These bounds should be compared with what is known for nondecaying traps, where a classical result of Donsker and Varadhan[6] says that in $d \geqslant 1$, if $\sum_x xp(x) = 0$ and $\sum_x x^i x^j p(x) = \sigma^2 \delta_{ij}$ with $0 < \sigma^2 < \infty$, then

$$\lim_{n \to \infty} n^{-d/(d+2)} \log f(n) = -K_d (\sigma^d \lambda)^{2/(d+2)}$$

with $\lambda = -\log(1-c)$ and $K_d > 0$ some constant depending on d (e.g., $K_1 = \frac{3}{2}\pi^{2/3}$). It is natural to guess that a similar result will hold for decaying traps as well, but we are far from being able to prove this. Theorem 2 identifies the exponent and the scaling dependence in $d = 1$. In Section 3 we give a *heuristic* argument suggesting that the exponent becomes

$$\begin{aligned} &(d-2\gamma)/(d+2), && 0 < \gamma < 2/d \\ &1-\gamma, && 2/d \leqslant \gamma < 1 \end{aligned}$$

Remarkably this exhibits a crossover at $\gamma = 2/d$.

Our third result concerns the behavior of $f(n)$ for moderately large n and small c with $d(n)$ fixed [recall $c(n)=cd(n)$]. This is obtained from a cumulant expansion of (1.3) and again parallels what is known for the non-decaying trap situation.[1,2] In fact, for small c one immediately sees that the first cumulant dominates, i.e.,

$$\log f(n) \sim -c \sum_{0 \leqslant k \leqslant n} d(k)\, E_X R(k) \qquad (c \to 0)$$

which can be evaluated using (1.5). The smaller is c, the larger may n be taken. As $c \to 0$, we may even insert the asymptotic form of $E_X R(k)$ for k large, since the sum diverges as $n \to \infty$ [recall (1.4)]. This will be written out in Section 2.4 and leads to the following result:

Theorem 3. Suppose that

$$\begin{cases} \sum_x xp(x) = 0 & d=1,2 \\ 0 < \sigma^2 = \det \left\{ \sum_x x^i x^j p(x) \right\} < \infty & \end{cases}$$

$$L = P_X(X_k \neq 0 \text{ for all } k > 0) \qquad d > 3$$

and that $c(n)=cd(n)$ with $d(n) \sim An^{-\gamma}$ ($n \to \infty$, $A > 0$, $0 < \gamma < 1/2$ for $d=1$, $0 < \gamma < 1$ for $d \geqslant 2$). Then for small c and n moderately large

$$\log f(n) \sim \begin{cases} -\left(\dfrac{2}{\pi}\right)^{1/2} \dfrac{2}{1-2\gamma}\, \sigma c A n^{1/2-\gamma} & d=1 \\[2ex] -2\pi \dfrac{1}{1-\gamma}\, \sigma c A\, \dfrac{n^{1-\gamma}}{\log n} & d=2 \\[2ex] -\dfrac{1}{1-\gamma}\, LcAn^{1-\gamma} & d \geqslant 3 \end{cases} \tag{1.7}$$

From (1.7) we can locate the regime of n values where $f(n)$ changes from $O(1)$ to $o(1)$ as $c \to 0$:

$$\begin{array}{ll} \left(\dfrac{1}{c}\right)^{2/(1-2\gamma)} & d=1 \\[2ex] \left(\dfrac{1}{c}\log\dfrac{1}{c}\right)^{1/(1-\gamma)} & d=2 \\[2ex] \left(\dfrac{1}{c}\right)^{1/(1-\gamma)} & d \geqslant 3 \end{array}$$

Our fourth and final result concerns the expected trapping time as $c \to 0$. To calculate $ET = \sum_{n \geqslant 0} f(n)$, we need to estimate the higher cumulants because we need to get an idea of how $f(n)$ behaves in the intermediate regime between moderately large n and large n. This turns out to be a hard problem because the random variables $R(k)$ are difficult to manipulate. For nondecaying traps it has been shown that the first cumulant is the dominating contribution to ET in the limit as $c \to 0$ in $d \geqslant 2$.(7) This comes from the fact that for small c the approximation of $f(n)$ by the first cumulant is quite good up to relatively large n. By the time the higher cumulants come into play, the function $f(n)$ has already made a substantial drop, so that the late terms in the sum $\sum_{n \geqslant 0} f(n)$ have a small contribution. It is reasonable to expect that something similar will occur for decaying traps as well, but we have trouble in getting the right estimates for the higher cumulants. Theorem 4 below is restricted to transient random walks, where the estimates turn out to be easier. All random walks in $d \geqslant 3$ are transient, as well as all those in $d = 1$ and 2 with $\sum_x xp(x) \neq 0$. In Section 2.5 we prove the following result.

Theorem 4. Suppose that the random walk is transient and that $c(n) = cd(n)$ with $d(n) \sim An^{-\gamma}$ ($n \to \infty$, $A > 0$, $0 < \gamma < 1$). Then

$$\lim_{c \to 0} c^{1/(1-\gamma)} ET = (1-\gamma)^{\gamma/(1-\gamma)}\, \Gamma\left(\frac{1}{1-\gamma}\right)\left(\frac{1}{AL}\right)^{1/(1-\gamma)} \tag{1.8}$$

where $L = P_X(X_k \neq 0 \text{ for all } k > 0) > 0$ and Γ is the gamma function.

This generalizes what is known for nondecaying traps[1,2,7]

$$\lim_{c \to 0} cET = L^{-1}$$

2. PROOFS

2.1. Proof of Proposition

From (1.1) and (1.2) we have

$$f(n) = P(C_k(X_k) = 0 \text{ for } 0 \leqslant k \leqslant n)$$

Writing this out with indicators, we get

$$\begin{aligned} f(n) &= E_X E_C \left(\prod_{0 \leqslant k \leqslant n} 1(C_k(X_k) = 0) \right) \\ &= E_X E_C \left(\prod_{0 \leqslant k \leqslant n} 1(\tau(X_k) \leqslant k) \right) \\ &= E_X E_C \left(\prod_{\{0 \leqslant k \leqslant n:\, R(k) = 1\}} 1(\tau(X_k) \leqslant k) \right) \end{aligned}$$

where in the last equality we use the fact that traps are only allowed to decay: if the walk hits an old site, then this site cannot be a trap, because already it was not a trap at each of the previous hits, otherwise the walk would not have survived. If we now use that the lifetimes at the distinct sites $\{X_k: R(k)=1\}$ are independent, then it follows that

$$f(n)=E_X\left(\prod_{\{0\leqslant k\leqslant n:\, R(k)=1\}}[1-c(k)]\right)$$

which is the same as (1.3). ∎

2.2. Proof of Theorem 1

From now on we shall drop the subscript X and use P and E to denote probability and expectation over the random walk. Note that (1.3) may be rewritten as

$$\begin{aligned} f(n) &= Ee^{-U(n)} \\ U(n) &= \sum_{0\leqslant k\leqslant n} \lambda(k)\, R(k) \end{aligned} \tag{2.1}$$

with $\lambda(k)=-\log[1-c(k)]$. Since $\lambda(k)\sim c(k)$ as $c(k)\to 0$, Theorem 1 amounts to proving the following result.

Lemma 1. $U(\infty)=\infty$ P-a.s. if and only if $EU(\infty)=\infty$.

Proof. Obviously $U(\infty)=\infty$ P-a.s. implies $EU(\infty)=\infty$, so we must show the reverse.

We start with the observation that $\{U(\infty)=\infty\}$ is a tail event of the random walk, i.e.,

$$\{U(\infty)=\infty\}\in\mathscr{T}=\bigcap_{N\geqslant 0}\sigma((X_n)_{n\geqslant N})$$

To see why, let $\omega=(\omega_n)$ and $\omega'=(\omega'_n)$ be two realizations of (X_n). Then a little thought shows that

$$|\{k\geqslant 0: R(k,\omega)\neq R(k,\omega')\}|\leqslant 2\,|\{n\geqslant 0: \omega_n\neq\omega'_n\}|$$

This may be checked by induction on n, because a change of ω at ω_n affects $R(k,\omega)$ at not more than two values of $k\geqslant n$. It follows that if $\omega_n=\omega'_n$ for $n\geqslant N$, then

$$|U(\infty,\omega)-U(\infty,\omega')|\leqslant 2N\lambda(0)$$

[because $\lambda(k)$ is decreasing], so either both are $<\infty$ or $=\infty$.

By the Hewitt–Savage zero–one law (ref. 8, p. 62–63), $\mathscr{T}$ is trivial and hence

$$P(U(\infty)=\infty)=0 \text{ or } 1$$

Thus, to complete the proof, it suffices to show that $EU(\infty)=\infty$ implies $P(U(\infty)=\infty)>0$.

First we show that

$$EU^2(n) \leqslant 2(EU(n))^2 \qquad \text{for all } n \tag{2.2}$$

Indeed, let

$$R(m, n)=1(X_n \notin \{X_m,..., X_{n-1}\}) \qquad (0 \leqslant m \leqslant n)$$

Write

$$EU(n)=\sum_{k=0}^{n} \lambda(k)\, ER(0, k)$$

$$\begin{aligned} EU^2(n) &= \sum_{k=0}^{n} \sum_{l=0}^{n} \lambda(k)\, \lambda(l)\, ER(0, k)\, R(0, l) \\ &\leqslant 2 \sum_{k=0}^{n} \sum_{l=k}^{n} \lambda(k)\, \lambda(l)\, ER(0, k)\, R(0, l) \end{aligned}$$

Note that for every $0 \leqslant k \leqslant l$

$$\begin{aligned} ER(0, k)\, R(0, l) &\leqslant ER(0, k)\, R(k, l) \\ &= ER(0, k)\, ER(k, l) \\ &= ER(0, k)\, ER(0, l-k) \end{aligned}$$

and $\lambda(k)\, \lambda(l) \leqslant \lambda(k)\, \lambda(l-k)$. Substitute to get (2.2).

Next we show that

$$P(U(n)/EU(n) > \tfrac{1}{2}) \geqslant \tfrac{1}{8} \qquad \text{for all } n \tag{2.3}$$

Indeed, by Cauchy–Schwarz,

$$\begin{aligned} \tfrac{1}{2}EU(n) &\leqslant E(U(n)\, 1(U(n) > \tfrac{1}{2}EU(n))) \\ &\leqslant \{EU^2(n)\, P(U(n) > \tfrac{1}{2}EU(n))\}^{1/2} \end{aligned}$$

which gives (2.3) via (2.2).

Finally, we let $n \to \infty$ in (2.3). If $EU(\infty)=\infty$, then $P(U(\infty)=\infty) \geqslant \frac{1}{8} > 0$. ∎

What is nice about Lemma 1 is that it holds for an *arbitrary* random walk. However, it should be noted that for most random walks Lemma 1 could also be obtained as a corollary of some known results for the range

$$S(n) = \sum_{0 \leqslant k \leqslant n} R(k) = |\{X_0, X_1, \ldots, X_n\}|$$

Namely, for transient random walk (ref. 5, p. 38) and for recurrent random walk in $d = 2$[9] the strong law $S(n)/ES(n) \to 1$ P-a.s. $(n \to \infty)$ holds, while for recurrent stable law random walks in $d = 1$[9,10] $S(n)/ES(n) \to Z$ in distribution $(n \to \infty)$ with Z some nondegenerate random variable. These results immediately yield Lemma 1 after we rewrite

$$U(n) = \sum_{0 \leqslant m \leqslant n} \Delta(m)\, S(m) + \lambda(n+1)\, S(n)$$

with $\Delta(m) = \lambda(m) - \lambda(m+1) \geqslant 0$.

But for recurrent random walk in $d = 1$ that is not stable very little is known about $S(n)/ES(n)$. In this connection it is interesting to put forward the following result, the proof of which will serve us later.

Lemma 2. $S(n)/ES(n)$ is tight from below, i.e., for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$P(S(n)/ES(n) > \delta) > 1 - \varepsilon \qquad \text{for all } n \tag{2.4}$$

Proof. First we note that (2.3) also applies to $S(n)$, i.e.,

$$P(S(n)/ES(n) > \tfrac{1}{2}) \geqslant \tfrac{1}{8} \qquad \text{for all } n \tag{2.5}$$

because $U(n) = \lambda(0)\, S(n)$ in the case of nondecaying traps.

Next we note that for every $\delta > 0$ and n such that δn is integer

$$S(n) \geqslant \max_{0 \leqslant j < \delta^{-1}} S^j(\delta n) \tag{2.6}$$

where

$$S^j(\delta n) = \sum_{k = j\,\delta n}^{(j+1)\,\delta n} R(j\,\delta n, k)$$

Now, $S^j(\delta n)$ are i.i.d. and distributed like $S(\delta n)$, so it follows from (2.5) that there exists $\varepsilon(\delta) \to 0$ as $\delta \to 0$ such that

$$P(\max_{0 \leqslant j < \delta^{-1}} S^j(\delta n) > \tfrac{1}{2} ES(\delta n)) > 1 - \varepsilon(\delta) \tag{2.7}$$

Finally, we note that

$$S(n) \leqslant \sum_{0 \leqslant j < \delta^{-1}} S^j(\delta n)$$

and hence $ES(n) \leqslant \delta^{-1} ES(\delta n)$. Combine the latter with (2.6) and (2.7) to get

$$P(S(n) \geqslant \tfrac{1}{2}\delta ES(n)) > 1 - \varepsilon(\delta)$$

which completes the proof of Lemma 2. ∎

Incidentally, note that by (2.2) and Chebyshev, $S(n)/ES(n)$ is also tight from above.

2.3. Proof of Theorem 2

Upper Bound. Because $c(n)$ is decreasing in n, we have from (2.1)

$$f(n) \leqslant E e^{-c(n)\, S(n)}$$

Let $h(n)$ be a positive function of n, which is to be specified later. Then

$$f(n) \leqslant P(S(n) \leqslant h(n)) + e^{-c(n)\, h(n)} \tag{2.8}$$

Lemma 3. For any $h(n)$ with $\lim_{n \to \infty} h(n) = \infty$,

$$P(S(n) \leqslant h(n)) \leqslant (n+1)\, e^{-\sigma^2 n/4h^2(n)} \qquad \text{for } n \text{ large} \tag{2.9}$$

Proof. Let

$$l(n; x) = \sum_{0 \leqslant k \leqslant n} 1(X_k = x) = \text{local time at site } x \text{ up to time } n$$

Since $\sum_x l(n; x) = n + 1$, it follows that

$$\begin{aligned} n + 1 &= \sum_x l(n; x)\, 1(X_k = x \text{ for some } 0 \leqslant k \leqslant n) \\ &\leqslant \{\sup_x l(n; x)\} \sum_x 1(X_k = x \text{ for some } 0 \leqslant k \leqslant n) \\ &= \{\sup_x l(n; x)\}\, S(n) \end{aligned}$$

and hence

$$P(S(n) \leqslant h(n)) \leqslant P(\sup_x l(n; x) \geqslant n/h(n))$$

Now

$$P(\sup_x l(n;x) \geqslant n/h(n))$$

$$\leqslant \sum_{0 \leqslant k \leqslant n} P(l(n;X_k) \geqslant n/h(n),\, X_k \notin \{X_0,..., X_{k-1}\})$$

$$\leqslant (n+1)\, P(l(n;0) \geqslant n/h(n))$$

where the latter inequality follows because $l(n;0)$ is stochastically larger than $l(n;x)$ for all x. So we have

$$P(S(n) \leqslant h(n)) \leqslant (n+1)\, P(l(n;0) \geqslant n/h(n)) \tag{2.10}$$

Next, let $(\rho_m)_{m \geqslant 1}$ be the successive times at which the walk returns to the origin. Then, for $n/h(n)$ integer,

$$P(l(n;0) \geqslant n/h(n)) = P(\rho_{n/h(n)} \leqslant n)$$

Since ρ_m is the sum of m i.i.d. copies of ρ_1, it follows from the Markov inequality that for any $\xi > 0$

$$P(\rho_{n/h(n)} \leqslant n) \leqslant e^{\xi n}\{Ee^{-\xi \rho_1}\}^{n/h(n)}$$

Moreover, we have the renewal relation[3,4]

$$Ee^{-\xi\rho_1} = \sum_{n \geqslant 1} e^{-\xi n} P(\rho_1 = n) = 1 - \left\{\sum_{n \geqslant 0} e^{-\xi n} P(X_n = 0)\right\}^{-1}$$

So far all steps in the argument are valid for an *arbitrary* random walk. Now, if $d = 1$, $\sum_x xp(x) = 0$ and $0 < \sigma^2 = \sum_x x^2 p(x) < \infty$, then[3,4]

$$\sum_{n \geqslant 0} e^{-\xi n} P(X_n = 0) \sim (2\sigma^2 \xi)^{-1/2} \qquad (\xi \to 0)$$

Hence,

$$Ee^{-\xi\rho_1} = e^{-(2\sigma^2\xi)^{1/2}[1 + o(1)]}$$

If we now choose ξ such that $\xi n - (2\sigma^2\xi)^{1/2} n/h(n)$ is minimal, i.e., $\xi = \sigma^2/2h^2(n)$, then we get that

$$P(\rho_{n/h(n)} \leqslant n) \leqslant e^{-\sigma^2 n/4h^2(n)} \qquad \text{for } n \text{ large}$$

Combine with (2.10) to get (2.9). ∎

If, finally, we choose $h(n)$ such that the two exponents in (2.8) and (2.9) become equal, i.e., $h(n) = \{\sigma^2 n/4c(n)\}^{1/3}$, then we end up with

$$f(n) \leqslant (n+2)\, e^{-\{\sigma^2 n c^2(n)/4\}^{1/3}} \qquad \text{for } n \text{ large}$$

which proves the upper bound in (1.6) with $K_1 < (1/4)^{1/3}$.

Lower Bound. The key is the following lemma.

Lemma 4. Let $D > 0$ and $0 < \delta < 1/2$. There exists $K > 0$ independent of σ, D, and δ such that

$$P(S(k) \leqslant Dk^\delta \text{ for } 0 \leqslant k \leqslant n) \geqslant e^{-[K\sigma^2/D^2(1-2\delta)]\, n^{1-2\delta}} \qquad \text{for } n \text{ large} \qquad (2.11)$$

Proof. The idea is to look at exit times of intervals. By the classical space-time scaling of random walk in $d = 1$ with zero mean and finite variance (ref. 5, p. 269) we have

$$P(\max_{0 \leqslant k \leqslant n} |X_k| \leqslant \tfrac{1}{2}\sigma n^{1/2}) \geqslant a \qquad \text{for } n \text{ large}$$

for some $a > 0$ independent of σ. We need a slight refinement of this property, in which we allow the starting point and the endpoint of the walk to vary over the middle half of the interval, namely

$$\inf_{|x| \leqslant 1/4(\sigma n^{1/2})} P^x(\max_{0 \leqslant k \leqslant n} |X_k| \leqslant \tfrac{1}{2}\sigma n^{1/2}, |X_n| \leqslant \tfrac{1}{4}\sigma n^{1/2}) \geqslant b \qquad \text{for } n \text{ large} \qquad (2.12)$$

for some $b > 0$ independent of σ (the upper index x means $X_0 = x$). This follows from a classical estimate for the Brownian bridge, via the invariance principle (ref. 8, p. 279).

Now let

$$b(j) = \left(\frac{D^2(1-2\delta)}{\sigma^2}\, j\right)^{1/(1-2\delta)}$$

and define the events

$$B(j) = \{|X_{b(j)}| \leqslant \tfrac{1}{4} D[b(j)]^\delta, |X_k| \leqslant \tfrac{1}{2} D[b(j)]^\delta \text{ for } b(j) \leqslant k \leqslant b(j+1)\}$$

Since

$$b(j+1) - b(j) \sim \frac{D^2}{\sigma^2}\,[b(j)]^{2\delta} \qquad (j \to \infty)$$

it follows from (2.12) that

$$P\left(B(j) \,\middle|\, \bigcap_{1 \leqslant i < j} B(i)\right) \geqslant b \qquad \text{for } j \text{ large} \qquad (2.13)$$

Pick $j(n)$ such that $b(j(n)) = n$. Then for n large

$$
\begin{aligned}
(\tfrac{1}{2}b)^{j(n)} &\leqslant P\Bigl(\bigcap_{1 \leqslant i < j(n)} B(i)\Bigr) \\
&\leqslant P(|X_k| \leqslant \tfrac{1}{2}Dk^\delta \text{ for } 0 \leqslant k \leqslant n) \\
&\leqslant P(S(k) \leqslant Dk^\delta \text{ for } 0 \leqslant k \leqslant n) \qquad (2.14)
\end{aligned}
$$

This proves (2.11) because

$$j(n) = \frac{\sigma^2}{D^2(1-2\delta)}\, n^{1-2\delta} \quad \blacksquare$$

From Lemma 4 we continue as follows. Return to (2.1). Since $c(n) \sim \lambda(n) \sim An^{-\gamma}$ we get from (2.11), provided $\delta > \gamma$,

$$
\begin{aligned}
\log f(n) &\geqslant -\Bigl\{\frac{K\sigma^2}{D^2(1-2\delta)}\, n^{1-2\delta} + [1+o(1)]\, AD \\
&\qquad \times \sum_{0<k\leqslant n} k^{-\gamma}[k^\delta - (k-1)^\delta]\Bigr\} \\
&\sim -\Bigl[\frac{K\sigma^2}{D^2(1-2\delta)}\, n^{1-2\delta} + \frac{AD\delta}{\delta-\gamma}\, n^{\delta-\gamma}\Bigr] \qquad (2.15)
\end{aligned}
$$

The exponent is minimal when

$$\delta = \frac{1}{3}(1+\gamma)$$

$$D = \left(\frac{6K\sigma^2}{A(1+\gamma)}\right)^{1/3}$$

This proves the lower bound in (1.6) with $K_2 > [\tfrac{9}{2}(1+\gamma)]^{2/3} K^{1/3}$. $\blacksquare$

2.4. Proof of Theorem 3

Theorem 3 amounts to nothing more than a calculation of the first cumulant $c \sum_{0 \leqslant k \leqslant n} d(k)\, ER(k)$. Indeed, (1.5) gives[3,4]

$$ER(n) \sim \begin{cases} (2\sigma^2/\pi n)^{1/2} & d=1 \\ 2\pi\sigma/\log n & d=2 \quad (n \to \infty) \\ L & d \geqslant 3 \end{cases}$$

with σ and L as defined in Theorem 3. Together with $d(n) \sim An^{-\gamma}$, this immediately gives (1.7). $\blacksquare$

2.5. Proof of Theorem 4

We have already seen that

$$\lim_{n \to \infty} ER(n) = L > 0 \tag{2.16}$$

where $L > 0$ by assumption of transience. Recall (2.1),

$$\begin{aligned} f(n) &= Ee^{-U(n)} \\ U(n) &= \sum_{0 \leqslant k \leqslant n} \lambda(k)\, R(k) \end{aligned} \tag{2.17}$$

with $\lambda(k) = -\log[1 - c(k)]$. Since $c(n) = cd(n)$ with $d(n) \sim An^{-\gamma}$, we have $\lambda(n) \sim Acn^{-\gamma}$ uniformly in c, and hence from (2.16) and (2.17)

$$EU(n) \sim \frac{AL}{1-\gamma}\, cn^{1-\gamma} \qquad \text{uniformly in } c \quad (n \to \infty) \tag{2.18}$$

Apply Jensen's inequality to (2.17) and substitute (2.18) to obtain, as $c \to 0$,

$$\begin{aligned} ET &= \sum_{n \geqslant 0} f(n) \geqslant \sum_{n \geqslant 0} e^{-EU(n)} \\ &\sim \int_0^\infty dn \exp\left(-\frac{AL}{1-\gamma}\, cn^{1-\gamma}\right) \\ &= \left(\frac{1}{ALc}\right)^{1/(1-\gamma)} (1-\gamma)^{\gamma/(1-\gamma)} \int_0^\infty dt\, e^{-t} t^{\gamma/(1-\gamma)} \end{aligned}$$

The integral equals $\Gamma(1/(1-\gamma))$ and so we have proved the lower half of (1.8).

The upper half is more subtle. Split ET into two parts,

$$ET = I_1(c, n) + I_2(c, n)$$

with

$$\begin{aligned} I_1(c, n) &= \sum_{0 \leqslant n \leqslant Nc^{-1/(1-\gamma)}} f(n) \\ I_2(c, n) &= \sum_{n > Nc^{-1/(1-\gamma)}} f(n) \end{aligned}$$

We shall show that

$$\lim_{N \to \infty} \limsup_{c \to 0} c^{1/(1-\gamma)} I_2(c, n) = 0 \tag{2.19}$$

$$\limsup_{N \to \infty} \limsup_{c \to 0} c^{1/(1-\gamma)} I_1(c, n) \leqslant \text{r.h.s. } (1.8) \tag{2.20}$$

which will complete the proof of Theorem 4.

Proof of (2.19). We start from the following lemma.

Lemma 5. There exists $K>0$ (depending on L) such that for any $h(n)$ with $\lim_{n\to\infty} h(n)=\infty$ and $h(n)=o(n)$,

$$P(S(n)\leqslant h(n))\leqslant e^{-Kn/h(n)} \qquad \text{for } n \text{ large} \tag{2.21}$$

Proof. By (2.16),

$$\lim_{n\to\infty} \frac{1}{n} ES(n)=L>0 \tag{2.22}$$

Return to (2.5) and (2.6), which are both true for an arbitrary random walk. From (2.5) and (2.22) we have

$$P(S(n)\leqslant \tfrac{1}{4}Ln)\leqslant \tfrac{7}{8} \qquad \text{for } n \text{ large}$$

Now use (2.6) with $\delta=4h(n)/Ln$ to get

$$\begin{aligned} P(S(n)\leqslant h(n)) &\leqslant P\left(\max_{0\leqslant j<Ln/4h(n)} S^j(4h(n)/L)\leqslant h(n)\right)\\ &=\{P(S(4h(n)/L)\leqslant h(n))\}^{Ln/4h(n)}\\ &\leqslant (\tfrac{7}{8})^{Ln/4h(n)} \qquad \text{for } n \text{ large} \end{aligned}$$

which is (2.21). ▮

From Lemma 5 we continue as follows. Return to (2.8). Pick $h(n)=(Kn/c(n))^{1/2}$ to get

$$f(n)\leqslant 2e^{-(Knc(n))^{1/2}} \qquad \text{for } n \text{ large}$$

Next substitute $c(n)=cd(n)$ with $d(n)\sim An^{-\gamma}$ to obtain for c sufficiently small

$$\begin{aligned} I_2(c,n) &\leqslant 2\int_{Nc^{-1/(1-\gamma)}}^{\infty} dn\, e^{-(KAcn^{1-\gamma})^{1/2}/2}\\ &=2c^{-1/(1-\gamma)}\int_N^{\infty} dt\, e^{-(KAt^{1-\gamma})^{1/2}/2} \end{aligned}$$

This proves (2.19) because the integral converges. ▮

Proof of (2.20). We start from the following lemma.

Lemma 6.

$$\lim_{n\to\infty} \frac{\operatorname{Var} U(n)}{[EU(n)]^2}=0 \qquad \text{uniformly in } c \tag{2.23}$$

Proof. Just as in the proof of (2.2), we have

$$\begin{aligned} EU^2(n) &= \sum_{0 \leqslant k, l \leqslant n} \lambda(k)\, \lambda(l)\, ER(k)\, R(l) \\ &\leqslant 2 \sum_{0 \leqslant k \leqslant l \leqslant n} \lambda(k)\, \lambda(l)\, ER(k)\, R(l) \\ &\leqslant 2 \sum_{0 \leqslant k \leqslant l \leqslant n} \lambda(k)\, \lambda(l)\, ER(k)\, ER(l-k) \end{aligned}$$

Substitute (2.16) and it easily follows that $EU^2(n) \sim [EU(n)]^2$ uniformly in c. ▌

From Lemma 6 we continue as follows. First note that by (2.18) we have for c sufficiently small independently of N

$$EU(n) \leqslant 2 \frac{AL}{1-\gamma} N^{1-\gamma} \qquad \text{for} \quad 0 \leqslant n \leqslant Nc^{-1/(1-\gamma)} \tag{2.24}$$

This bound is independent of c. Now we claim that for every fixed N there exists $\varepsilon_N(c) \to 0$ as $c \to 0$ such that

$$E(e^{-(U(n) - EU(n))}) \leqslant e^{\varepsilon_N(c)\, EU(n)} \qquad \text{for} \quad \log \frac{1}{c} \leqslant n \leqslant Nc^{-1/(1-\gamma)} \tag{2.25}$$

[instead of $\log(1/c)$ pick any function tending to ∞ as $c \to 0$]. Indeed, the trivial bound $0 \leqslant R(k) \leqslant 1$ together with (2.16) and (2.17) implies that there exists $K > 0$ (depending on L) such that

$$|U(n) - EU(n)| \leqslant KEU(n) \qquad \text{for all } n \text{ and } c$$

[pick $K = \max\{1, L^{-1} - 1\}$ because $ER(n)$ is decreasing in n]. Hence

$$\begin{aligned} E\,|U(n) - EU(n)|^k &\leqslant [KEU(n)]^{k-2} \operatorname{Var} U(n) \\ &= [KEU(n)]^k\, \delta_n(c) \qquad (k \geqslant 2) \end{aligned}$$

for some $\delta_n(c) \to 0$ as $n \to \infty$ uniformly in c because of (2.23). The latter immediately gives (2.25) via (2.24). Now proceed from (2.25) by estimating

$$\begin{aligned} I_1(c, n) - \log \frac{1}{c} &\leqslant \sum_{\log(1/c) \leqslant n \leqslant Nc^{-1/(1-\gamma)}} e^{-(1-\varepsilon_N(c))\, EU(n)} \\ &\sim \int_{\log(1/c)}^{Nc^{-1/(1-\gamma)}} dn\, e^{-(1-\varepsilon_N(c)) ALcn^{1-\gamma}/(1-\gamma)} \\ &= \left(\frac{1}{ALc}\right)^{1/(1-\gamma)} (1-\gamma)^{\gamma/(1-\gamma)} \\ &\quad \times \int_{ALc[\log(1/c)]^{1-\gamma}/(1-\gamma)}^{ALN^{1-\gamma}/(1-\gamma)} dt\, t^{\gamma/(1-\gamma)} e^{-(1-\varepsilon_N(c))t} \end{aligned}$$

Let first $c \to 0$ and then $N \to \infty$ to recover (2.20). ▌

3. HEURISTIC ARGUMENT FOR $d \geqslant 2$

In this section we give a *heuristic* argument in favor of the exponent in higher dimension conjectured below Theorem 2. This parallels a similar argument for nondecaying traps (ref. 2, p. 381).

The main idea is to do a *large-deviation* estimate as follows. The probability that the random walk survives trapping during n steps, given that its range is $(S(m))_{m=0}^{n}$ over the time interval $0 \leqslant m \leqslant n$, equals

$$\exp\left\{-\left[\sum_{0 \leqslant m \leqslant n} \Delta(m)\, S(m) + \lambda(n+1)\, S(n)\right]\right\} \tag{3.1}$$

with $\Delta(m) = \lambda(m) - \lambda(m+1)$. Since this probability is large when $(S(m))_{m=0}^{n}$ is small, we expect that the dominant contribution to survival comes from walks for which $S(m)$ grows slower than is typical (which is order $m/\log m$ in $d=2$ and m in $d \geqslant 3$). We therefore pick $S(m) \approx Bm^{\delta}$ ($B>0$, $0<\delta<1$), where B and δ are parameters that may be varied.

We now want to estimate the probability that the random walk realizes $S(m) \approx Bm^{\delta}$ for $0 \leqslant m \leqslant n$. In order to do so, it must return to old sites many times [since $S(m+1) = S(m)$ for most $0 \leqslant m < n$ when $\delta < 1$], and therefore we expect it to curl up and roughly fill a sphere of radius $[S(m)]^{1/d}$ up to time m (we ignore numerical factors). In addition, at time m we expect it to be located in this sphere roughly according to a Gaussian with variance $[S(m)]^{2/d}$. This allows us now to estimate the probability that at time $m+1$ the walker does not exit the sphere [i.e., $S(m+1) = S(m)$], namely

$$\exp\{-C[S(m)]^{-2/d}\} \qquad (C>0)$$

Indeed, $[S(m)]^{2/d}$ is the average time needed to exit the sphere starting from somewhere in the middle. Thus, the probability of realizing the given $(S(m))_{m=0}^{n}$ equals

$$\exp\left\{-C \sum_{0 \leqslant m \leqslant n} [S(m)]^{-2/d}\right\} \tag{3.2}$$

where we estimate that the events $S(m+1) = S(m)$ for successive m are roughly independent and that the events $S(m+1) \neq S(m)$ have a negligible contribution.

By combining (3.1) and (3.2), substituting $\lambda(n+1) \approx c(n+1) \approx An^{-\gamma}$, $\Delta(m) \approx \gamma A m^{-\gamma-1}$, and $S(m) \approx Bm^{\delta}$, we find for large n

$$\log f(n) \approx \sup_{B,\delta}\left\{-\frac{\delta}{\delta-\gamma} ABn^{\delta-\gamma} - \frac{C}{(1-2\delta/d)\, B^{2/d}} n^{1-2\delta/d}\right\} \tag{3.3}$$

The exponent is maximal when $\delta - \gamma = 1 - 2\delta/d$, i.e., $\delta = d(1+\gamma)/(d+2)$, which yields the conjectured exponent $(d-2\gamma)/(d+2)$.

The argument breaks down when $\delta = d(1+\gamma)/(d+2) > 1$ because $S(n)$ can never exceed n. This explains the conjectured crossover at $\gamma = 2/d$. The supremum is then attained at $\delta = 1$, and the exponent is $1-\gamma$.

The above argument is heuristic because it relies on the assumption that the dominant contribution is the one described. At best the argument can be turned into a rigorous lower bound for $f(n)$.

4. DISCUSSION

Theorem 1 is a nice result because it is completely general. Perhaps the condition (1.4) is somewhat surprising, because it is *not* true in general that $\log f(n) \sim -\sum_{k=0}^{n} c(k)\, E_X R(k)$ as $n \to \infty$ [first cumulant of (1.3)]. This can be seen, for instance, by comparing Theorems 2 and 3. Rather, the first cumulant is the moderately large-n approximation (see the remark prior to Theorem 3).

Theorem 2 obviously calls for more. How to narrow down the constants? How to patch up the heuristic argument of Section 3 and prove the conjectured exponent for $d \geqslant 2$? Another interesting question is what happens in $d = 1$ at the critical value $\gamma = 1/2$ [by Theorem 1, if $\gamma > 1/2$, then $f(\infty) > 0$]. Does $f(n)$ decay as a power of n?

Theorems 3 and 4 are fairly general. What is the analogue of Theorem 4 in $d = 1$ and 2? What happens at the critical value $\gamma = 1$ in the transient case?

There are some potentially interesting applications of our model. For instance, the walk may represent the diffusive motion of a particle of some chemical substance, and the decaying traps some chemical reactant that either disappears spontaneously according to some intrinsic mechanism or is destroyed by some outside agency (e.g., annealing, diffusive bleaching, radiation). A less obvious—but more amusing—application is to human lifetime statistics. View life as a "random walk" in a "decision space," with months or years corresponding to one unit time step. View the various potentially fatal diseases (e.g., pneumonia, cancer, AIDS) as "traps." Medical advances are progressively effecting cures for many formerly deadly illnesses. Thus, a human being will be exposed during the aging process to a decreasing number of diseases, with a consequent increase in survival. In the medical literature attempts have been made to describe the observed steady increase in life expectancy via a model of the type discussed here.[11]

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